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EQUATIONS AND VARIABLES ASSOCIATED WITH THE LINEAR DIFFERENTIAL EQUATION.

By Dr. George F. Metzler, Kingston, Ont., Canada.

Second Article.—Geometrical Interpretations.

Any one wishing to read on the geometry of space higher than three dimensions is referred to an article by G. Veronese published in the Mathematische Annalen, Bd. XIX, 1882, entitled "Behandlung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen durch das Princip des Projicirens und Schneidens," also to a book on the subject by the same author. Another helpful contribution in Italian is "Studio sulle quadriche in uno spazio linare ad un numero quelnuque di dimensione," by Corrado Segre (Memoire della Realla Accademia Delle Scienze di Torino, Tomo. XXXVI, 1885).

A few definitions and theorems relative to such a space will be introduced here:

Consider a space of $n-1$ dimensions S_{n-1}, the homogeneous coordinates of an element of which are $x_1, x_2, x_3, \ldots, x_n$.

In a space of $n-1$ dimensions, k relations between the n homogeneous coordinates represent a surface in $n-1-k$ dimensions S_{n-1-k}. If $k = n-2$, the coordinates are functions of a single parameter and we have a curve S_1. If $k = n-1$ we have a limited number of points S_0.

Every element of a linear space S'_{n-1} can be determined if we take in place of the $x_1, x_2, \ldots, x_n$, as coordinates $y_1, y_2, \ldots, y_n$, where the y's are proportional to given functions of the x's, which are linear, homogeneous, and independent; that is

$$\rho y_i = \sum_1^n l_{ik} x_k .$$

Whatever be the nature of the elements of this space they are called points.

A plane is a surface in which all the k relations are linear. A plane S_1' is a straight line, and S_0' a unique point. Instead of considering the surfaces as the locus of points, we can consider them as the envelope of planes S'_{n-2}. We classify them, then, according to the number of parameters on which the tangent planes S'_{n-2} depend. When S'_{n-2} depends on a single parameter it envelopes a developable $\triangle$.

When $m \leq n$, every space of $m - 1$ dimensions S_{m-1} will be contained in S_{n-1}, and among those notably the linear spaces S'_{m-1}. S'_{m-1} will be linear in the sense intended here when it consists of all those points whose coordinates x_i considered in the ultimate space S_{n-1} are given linear homogeneous functions of the m y's; that is

$$x_i = \sum_1^m l_{ik} y_k .$$

From this it follows that in order that the x be coordinates of the points of a space linear in $m - 1$ dimensions, it is necessary and sufficient that they satisfy certain $n - m$ linear homogeneous equations.

In general, a space S'_{n-1-l} is composed of those points whose coordinates x satisfy l linear homogeneous equations, and hence the ∞^{l-1} equations formed by combining these l equations linearly, or of points whose coordinates can be represented as linear homogeneous functions of $n - l$ other quantities (variable coordinates of the points in S'_{n-1-l}). S'_{n-1-l} is then the intersection of ∞^{l-1} S'_{n-2}.

A space S'_{n-2} is particularized by its equation

$$\xi_1 x_1 + \xi_2 x_2 + \ldots - \xi_n x_n = 0$$

in which the ξ are n given coefficients. Given $n - 1$ such equations, they serve to determine the ratio of the ξ.

Thus the S_{n-2} of points in S'_{n-1} can in their turn serve as the elements of a space Σ, also linear and of $n - 1$ dimensions. The ξ will be the homogeneous coordinates of a S'_{n-2} considered as elements of Σ.

An arbitrary linear homogeneous equation between the coordinates ξ of a S'_{n-2} necessitates that this pass through a fixed point, of which the coordinates are the coefficients of the ξ. Let

$$x_1\xi_1 + x_2\xi_2 + \ldots + x_n\xi_n = 0 \qquad (1)$$

be this equation; it may be called the equation of the point x_i in plane coordinates ξ_i.

From (1) we see that when the ξ's are constant we have a plane S'_{n-2} in S_n which is an element of Σ; and when the x's are constant we have a Σ'_{n-2} in Σ which is an element of S. Through this duality a proposition proven for one yields another for the other by an interchange of planes S'_{n-2} and points S'_0. Σ'_m will denote a linear space of m dimensions in Σ. Its elements are planes S'_{n-2} with the ξ for coordinates.

Every S'_{n-k} is the intersection common to a Σ'_{k-2}. Two such linear spaces are called conjugate. Two linear spaces S'_{n-k} and S'_{k-2}, the x alone being variable, are called dual.

If we have given $k+1$ points $x_i', x_i'', x_i''', \ldots, x_i^{(k+1)}$ $(i = 1, 2, \ldots, n)$ of a S_k' it is determined and any other of its points has the coordinates

$$x_i = h'x_i' + h''x_i'' + h'''x_i''' + \ldots + h^{(k+1)}x_i^{(k+1)},$$

where the $h^{(\nu)}$ can be considered as the $k+1$ homogeneous coordinates of a point in a space of k dimensions S_k'. Here the accents affixed to h and x serve merely to distinguish different quantities and have no necessary connection with differentiation. Similarly, $k+1$ elements S'_{n-2} with coordinates $u_i', u_i'', \ldots, u_i^{(k+1)}$ $(i = 1, 2, \ldots, n)$ determine a Σ_k'. The coordinates of any element S'_{n-2} in Σ_k' can be represented by

$$u_i = g'u_i' + g''u_i'' + \ldots + g^{(k+1)}u_i^{(k+1)}.$$

The g's may serve as the coordinates of an arbitrary plane S'_{n-2} in Σ_k'.

As we can take all the planes S'_{n-2} of S as elements of a space different from S, so whatever k may be, if $< n-1$, we may take all the S_k' as elements of another space. If S_k' is determined by the points $x_i, x_i'', \ldots, x_i^{(k+1)}$ Clebsch has shown* that we can take as coordinates of S_k' the determinants of order $k+1$ formed from the matrix

$$\begin{vmatrix} x_1' & x_2' & \ldots & x_n' \\ x_1'' & x_2'' & \ldots & x_n'' \\ \ldots & \ldots & \ldots & \ldots \\ x_1^{(k+1)} & x_2^{(k+1)} & \ldots & x_n^{(k+1)} \end{vmatrix},$$

and analogously, we can determine the coordinates of a Σ' from the $k+1$ S'_{n-2} elements which determine Σ_k'.

The coordinates $u_1, u_2, \ldots, u_n$ of a plane S'_{n-2} are, therefore, the n first minors of the determinant Δ of the nth order,

$$\Delta = \begin{vmatrix} x_1' & x_2' & \ldots & x_n' \\ x_1'' & x_2'' & \ldots & x_n'' \\ \ldots & \ldots & \ldots & \ldots \\ x_1^{(n)} & x_2^{(n)} & \ldots & x_n^{(n)} \end{vmatrix} \equiv (x_1'x_2'' \ldots x_n^{(n)}).$$

When we consider a space Σ'_{n-2-k} conjugate to S_k' it will be determined

* Ueber die Fundamentalaufgabe der Invarianttheorie. (Abhandlungen der Kön. Gesellschaft der Wissenchafter zu Göttingen, Band XVII, 1872.)

by $n-1-k$ planes S'_{n-2} with coordinates u_i', u_i'', $\ldots$, $u_i^{(n-1-k)}$ $(i = 1, 2, \ldots, n)$. The coordinates of Σ''_{n-2-k} will be the minors of the matrix

$$\begin{vmatrix} u_1' & u_2' & \ldots & u_n' \\ u_1'' & u_2'' & \ldots & u_n'' \\ \ldots & \ldots & \ldots & \ldots \\ u_1^{(n-1-k)} & u_2^{(n-1-k)} & \ldots & u_n^{(n-1-k)} \end{vmatrix}$$

and these by a well known theorem in determinants are proportional to the complementary minors of Δ, that is,

$$(u_1' u_2'' \ldots u_{n-1-k}^{(n-1-k)}) = \Delta^{n-2-k}(x_{n-k}^{(n-k)} \ldots x_n^{(n)}) .$$

Thus the coordinates of two conjugates spaces are proportional.

Considering determinants similar to

$$\begin{vmatrix} x_1' & x_2' & \ldots & x_n' \\ x_1^{(k+1)} & x_2^{(k+1)} & \ldots & x_n^{(k+1)} \\ x_1' & x_2' & \ldots & x_n' \\ x_1^{(k+1)} & x_2^{(k+1)} & \ldots & x_n^{(k+1)} \end{vmatrix}, \qquad k < n-2.$$

it is easy to see that these coordinates are connected by relations and are, therefore, not all independent.

Two planes S_m' S_r' are determined respectively by $m+1$ and $r+1$ points. These points when $m + r \leqq n - 2$ will determine a S'_{m+r+1} containing S_m' and S_r'. S_m' and S_r' have, in general, no point common. When they have a S_a', that is, $a+1$ points common, there remain only $m + r + 1 - a$ distinct points which determine a S'_{m+r-a}. Vice versa, if through S_m' S_r' we can pass a S'_{m+r-a}, then for arbitrary m and r S_m' and S_r' will intersect in a S_a'.

When $S'_{m+r-a} \equiv S'_{n-1}$ then $m + r - n + 1 = a$. If $a = 0$, S_m' and S_r' have one point common; if $a < 0$, they have no points common. If $a = n$, then S_m' contains S_r'; when $m > r$, $a \leqq r$; and when $n + r < n - 1$ and S_m', S_r' have points common, then they are contained in a space of dimensions less than $n - 1$.

When a point moves according to some algebraic law in only two directions it describes a curve S_1. This is of the mth order if it is cut by a S_{n-2} in m points. A curve of the mth order S_1^m can lie only in an S_2, S_3, or S_m; for, were it contained in a S_{m+1}, and not in a lower space S_m, then we could cut it

by a S_m, which can meet it in only m points, and through these m points and another point of the curve pass a S_m. This S_m will cut the curve in $m + 1$ points, which is impossible.

A space S_m is called an algebraic space of order p when every linear space S'_{n-1-m} has with it, in general, p common points. It is indicated by S_m^p. Similarly, Σ_m^p will denote an algebraic space of class p, when any Σ'_{n-1-m} has in common with this Σ_m^p p planes S'_{n-2}, that is, that pass through an S'_{m-1}. An S_{n-2} is a surface, an S'_{n-2} is a surface of the first order. A Σ_{n-2} is an envelope, a Σ'_{n-2} is an envelope of the first class. An equation of degree g in x coordinates will represent a surface of order g, while the same in u coordinates will represent a surface of class g. r equations of degrees $p_1, p_2, \ldots, p_r$ in x coordinates determine a space S_{n-1-r}^k, in which the order k equals the product $p_1, p_2, \ldots, p_r$. If the coordinates were u we should have a $\Sigma_{n-1-r}^{p_1, p_2, \ldots, p_r}$ of class $p_1, p_2, \ldots, p_r$.

The tangents of a curve generate a two-dimensional developable surface; the osculating planes are tangent to this surface, and the osculating spaces of higher orders can be called osculating spaces of this two-dimensional surface.

The osculating planes S_2' describe a three-dimensional developable surface D_3. The osculating spaces S_3 of the curve are tangential spaces of its three-dimensional developable, etc.

Two algebraic spaces $S_m^r S_m^p$ do not in general intersect when $m + m' < n - 1$. When $m + m' \geqq n - 1$ they intersect in a $S_{m+m'-n+1}^{rp}$, and so on for a greater number of spaces.

An S_m^g has only in common with S_k' an $S_{m+k+1-n}^g$. If S_k' has still a point outside $S_{m+k+1-n}^g$, it will intersect S_m^g in a space of at least one dimension more, that is, an $S_{m+k+2-n}$. When $n > i > m$, if S_m^g be contained in S_i, then every S'_{n-1-m} will intersect S_m^g in g points contained in S'_{i-m} in which it intersects S_i'. Vice versa, if every S'_{n-1-m} intersects S_m^g in g points of an S'_{i-m}, that is, in an S_0^g of S'_{i-m}, then every S'_{n-m} intersects S_m^g in an S_1^g of an S'_{i-m+1}, and every S'_{n-m+1} intersects S_m^g in an S_2^g of an S'_{i-m+2}; finally every S'_{n-1} intersects S_m^g in an S_m^g of an S_i', that is S_m^g is contained in S_i'. Then the conditions necessary and sufficient in order that S_m^g may be contained in S_i' is that every S'_{n-1-m} intersect S_m^g in g points of an S'_{i-m}. g points are always contained in an S'_{g-1}, and the condition will always be satisfied when $i - m \geqq g - 1$ or $i \geqq m + g - 1$. Then it follows, that every S_m^g is always contained in a linear space of $m + g - 1$ dimensions (but can also lie in a linear space of p dimensions where $m < p < m + g$).

Every S_m^2 is contained in an S'_{m+1}.

Every S_1^g is contained in an S_v', where $v \leqq g$.

Hence the S_1^g can be distinguished in $g - 1$ species, according as they lie

in an S_g', or in an S'_{g-1}, or in an S'_{g-r}, etc., or finally in an S_2'. Thus there are two species of S_1^3 (curves of the third order), those which lie in an S_3' and those which lie in an S_2'.

It will not be difficult to apply this to the particular case in which the n homogeneous coordinates of a point in S_{n-1} are $y_1^{(v)}, y_2^{(v)}, \ldots, y_n^{(v)}$, solutions of a linear differential equation of the nth order. $v = 0, 1, 2, \ldots$, etc., these upper marks denoting differentiation as in my first paper,

$$u_1y_1 + u_2y_2 + \ldots + u_ny_n = 0 \tag{2}$$

is the equation of a plane S'_{n-2}. The u being functions of a single variable x, this plane will envelope a developable surface D_{n-1}, to form which we must add the equation

$$u_1'y_1 + u_2'y_2 + \ldots + u_n'y_n = 0\,. \tag{3}$$

When we also add

$$u_1''y_1 + u_2''y_2 + \ldots + u_n''y_n = 0 \tag{4}$$

we obtain an S'_{n-4}, which will envelope a D_{n-3}. The S'_{n-2}'s osculate D_{n-3}, that is, contain three successive S'_{n-4}.

Adding further

$$u_1'''y_1 + u_2'''y_2 + \ldots + u_n'''y_n = 0\,, \tag{5}$$

we have an S'_{n-5} enveloping a D_{n-4}, and so on until we arrive at

$$u_1^{(n-2)}y_1 + u_2^{(n-2)}y_2 + \ldots + u^{(n-2)}y_n = 0 \tag{n}$$

k being equal to any of the numbers $1, 2, 3, \ldots, n-1$, the k equations (2), (3), (4), $\ldots$, $(k+1)$ represent an S'_{n-k-1} of which the surface enveloped is a D_{n-k}. The $n-1$ equations represent a point of which the locus is a curve called the edge of regression of all the developables considered.

When we consider the S'_{n-2}'s as elements of a space Σ'_{n-1}, the u's being the coordinates of these elements, we have the $n-1$ equations

$$y_1^{(k)}u_1 + y_2^{(k)}u_2 + \ldots + y_n^{(k)}u_n = 0\,, \quad k = 0, 1, 2, \ldots, n-2.$$

and a series of propositions similar to the above.

When the y's are taken as the homogeneous coordinates of a point in S_{n-1}, as x varies this point describes a curve Γ belonging to S_{n-1} (Halphen's attached curve). Similarly, the point of which the u's are coordinates will describe a curve C.

To the points of the curve C correspond the tangents S'_{n-2} of the developable D_{n-1}, or the S'_{n-k} osculating the curve Γ, and to the points of Γ correspond Σ'_{n-k} osculating C. The S'_{n-k-1} osculating the one curve correspond to the Σ'_{k-1} osculating the other ($k = 1, 2, \ldots, n-1$).

The curve I' is not altered by the substitutions

$$z = f(x). \qquad v = y\,\frac{1}{\varphi(x)}.$$

Thus to a single curve belong an infinite number of equations.

For a single value of x, y can have an infinite number of values, all, however, expressible as linear combinations of the primitive values.

If we imagine x extended over a surface (a plane, or a Rieman's surface) on which the coefficients of the differential equation are one-valued, then to every closed path on this surface will correspond a certain linear substitution with constant coefficients

$$y_i' = \sum_1^n{}_k\, a_{ik} y_k .$$

To the totality of all the closed paths that can be drawn on this surface will correspond a certain group of linear substitutions of the y_i (monodromic group) which will be discontinuous. Thus I' is determined only so far as it is one of a system derived from each other by homographic substitutions.

Thus we have a collineation of S'_{n-1}. Every branch of the functions y_i is changed into another branch of the same functions.

It has been shown* that when a linear differential equation is self-adjoint and of odd order, there always exists a homogeneous relation between the fundamental solutions y_i, $\varphi(y) = 0$; that is, the curve I' is situated on a surface of the second degree represented by $\varphi(y) = 0$.

The equations

$$u_1^{(k)} y_1^{(l)} + u_2^{(k)} y_2^{(l)} + \ldots + u_n^{(k)} y_n^{(l)} = 0$$

$$(k = 0, 1, 2, \ldots, n-2;\ l = 0, 1, 2, \ldots, n-2;\ k + l \geqq n-1.)$$

show that the points y_i', y_i'', y_i''', etc. $(i = 1, 2, \ldots, n)$ lie on the surface $\varphi(y) = 0$, and also on the tangent at a point distinct from the point of contact. The tangent is then entirely situated on the surface.

As $y_1'', y_2'', \ldots, y_n''$ are points on the osculating surface S_2', and $y_1^{(k)}, y_2^{(k)}, \ldots, y_n^{(k)}$ are points on the osculating surface S_k' $(k = 2, 3, 4, \ldots)$, we see that these surfaces osculating the curve I' lie entirely on $\varphi(y) = 0$. I' is then an asymptotic line of the surface.

When n is even we consider the complex

$$\sum_i \left(\sum_k a_{ik} y'_k\right) y_i = 0 .$$

We see from this that the surface corresponding to the point y_k' in a cor-

* Vol. IX, No. 6, p. 173.

relative transformation with respect to the complex contains the points $y_k^{(l)}$ $\begin{bmatrix} k = 1, 2, \ldots, n \\ l = 0, 1, \ldots, n-2 \end{bmatrix}$, that is, is an osculating plane of the curve at the point considered. We have then the geometrical problem, To find curves such that their osculating planes in any point are the planes corresponding to this point in the complex.

The y's being homogeneous coordinates of the points on the curve Γ, and the u's the homogeneous coordinates of the points on C, it is seen that these curves correspond dually to one another.

We do not know that the equations corresponding to these two curves are adjoint, but we can say that by multiplying the y's by some factor they become adjoint.

When $A = 0$ is self-adjoint then the curve C must belong to the same system as Γ. This defines self-adjointness geometrically.

The following is given as an illustration: Let $n = 5$; then $\varphi(y) = 0$ becomes

$$2y_1'y_5 + 2y_2y_4 - y_3^2 = 0,$$

which may be written

$$y_1y_5 + y_2y_4 - y_3y_3 + y_4y_2 + y_5y_1 = 0; \tag{1}$$

then

$$y_1y_5' + y_2y_4' - y_3y_3' + y_4y_2' + y_5y_1' = 0, \tag{2}$$

$$y_1y_5'' + y_2y_4'' - y_3y_3'' + y_4y_2'' + y_5y_1'' = 0, \tag{3}$$

$$y_1y_5''' + y_2y_4''' - y_3y_3''' + y_4y_2''' + y_5y_1''' = 0, \tag{4}$$

$$y_1'y_5' + y_2'y_4' - y_3'y_3' + y_4'y_2' + y_5'y_1' = 0 = \varphi(y'),$$

$$y_1'y_5^{\text{iv}} + y_2'y_4^{\text{iv}} - y_3'y_3^{\text{iv}} + y_4'y_2^{\text{iv}} + y_5'y_1^{\text{iv}} = 0,$$

$$y_1'y_5'' + y_2'y_4'' - y_3'y_3'' + y_4'y_2'' + y_5'y_1'' = 0.$$

From these equations we see

a) The curve Γ lies on $\varphi(y) = 0$;

b) The tangent plane at y contains y', y'', and y''', and therefore osculates the curve.

$\varphi(y) = 0$ has the generating line S_1', viz:

$$y_1 + by_3 - cy_4 = 0,$$

$$y_2 - ay_3 + cy_5 = 0,$$

$$y_3 - ay_4 + by_5 = 0,$$

where a, b, and c are arbitrary functions of x. This generates the surface from the point of view of linear generators, while the plane S_2' generates it as

an enveloping tangent. Substituting the values of y_1, y_2, y_3 in the polar plane of the point u, and equating the coefficients of y_4 and y_5, S_2' is

$$u_1 + bu_3 - cu_4 = 0\,,$$

$$u_2 - au_3 + cu_5 = 0\,.$$

The intersection of S_1' with the next consecutive generator is given by

$$y_3 \frac{db}{dx} = y_4 \frac{dc}{dx}, \quad y_3 \frac{da}{dx} = y_5 \frac{dc}{dx}.$$

From which we have

$$\frac{y_3}{\frac{dc}{dx}} = \frac{y_4}{\frac{db}{dx}} = \frac{y_5}{\frac{da}{dx}} = t, \text{ say,}$$

where t is arbitrary.

Then it follows that

$$y_1 = cy_4 - by_3 = tc\frac{db}{dx} - tb\frac{dc}{dx},$$

$$y_2 = ay_3 - cy_5 = t\left[a\frac{dc}{dx} - c\frac{da}{dx}\right],$$

$$y_3 = t\frac{dc}{dx},$$

$$y_4 = t\frac{db}{dx},$$

$$y_5 = t\frac{da}{dx}.$$

If these values are to satisfy $\varphi(y) = 0$, the exponent of x in c must be made to depend on those of a and b.

A GENERAL THEOREM RELATING TO TRANSVERSALS, AND ITS CONSEQUENCES.

By Mr. A. L. Candy, Lincoln, Nebraska.

[Continued from Vol. X, p. 192.]

It is evident that some of the properties expressed in the preceding theorems are projective and therefore true for all conics. And it can easily be shown that a much larger number of these properties remain true when subjected to the process of "orthogonal" projection.

First, let us suppose all of the preceding figures to be projected orthogonally upon any plane. The "given circle" will project into a "given ellipse" and all other circles will project into "similar ellipses with axes parallel to the axes of the given ellipse," and straight lines into straight lines.

Since the orthogonal projection of a finite line upon any plane is equal to the line multiplied by the cosine of the angle which the line makes with the plane, it follows that if each of the segments, made by the sides of the quadrilateral and this system of ellipses upon any transversal through a D-point, be multiplied by the cosecant of the angle which this line makes with the projected transversal, the results may be substituted in any of the foregoing equations. If equations (8), (10), (12), (14), (18), (20), (23), (24), (24a), (27), (28), (35), (36) be treated in this way it will be found that in any equation this constant factor appears to the same degree in every term, and hence the new equation is of the same form as the original.

∴ These relations are also true in the projected figure. Therefore, in order to state a particular case of the corresponding theorems for a quadrilateral inscribed in an ellipse, it is only necessary to substitute "given ellipse" for "given circle" and in all other places for "circles" to write "similar ellipses with axes parallel to the axes of the given ellipse;" and in (5), of Theor. I, to omit the clause concerning the radius of the foci locus, and also to strike out the last statement of (2) in Theor. III. All the other conclusions are still valid.

The results obtained by conical projection are not so simple. Here it is necessary to observe that all circles pass through the same two points at infinity. Therefore these circles must project into a system of conics through two fixed points, and each conic otherwise fulfils the same conditions as the circle of which it is the projection. Thus, we derive the following for conics in general:—

If a quadrilateral is inscribed in a conic and on each of its sides a conic is described passing through a D-point and all intersecting the given conic in the same two points, and any right line is drawn through this D-point :

We have for conclusions (4), (5) (omitting the last clause), (7), (8) of Theor. I, and (2), (3) of Theor. II, by substituting for " circles," " conics intersecting the given conic in the same two points."

For (1) of Theor. II, we must write: " In a crossed quadrilateral three of these conics meet their respective opposite sides in six points which lie on another conic passing through the same two common points."

The projection of Theor. III and Fig. 8 requires a separate and fuller statement. The circles project into conics intersecting the given conic in the same two points as before.

Since the center of a circle is the pole of the line at infinity, the center of the given circle projects into the pole of the common chord with respect to the given conic, and similarly the center of each of the other circles projects into the pole of the same line with respect to the conic into which that circle projects.

We thus obtain the following theorem :—

If a triangle is self-polar with respect to a given conic, and if conics are described on each of its sides intersecting the given conic in two common points, the pole of the common chord for each of these conics lying on the side of the self-polar triangle upon which the conic is described ; and if three other conics are drawn intersecting the given conic in the same two points, and passing through the pole of this common chord with respect to the given conic, and each passing through one vertex of the self-polar triangle and having the pole of the common chord with respect to itself lying on the right line joining this vertex to the pole of the common chord with respect to the given conic, then :

1. Every straight line drawn through the pole of the common chord with respect to the given conic is cut by these six conics and the three sides of the self-polar triangle in an involution of which this fixed pole is the center.

2. The given conic is the locus of the foci of this involution as the transversal is supposed to revolve about this fixed pole.

3. Every chord intercepted on this variable transversal by the given conic is divided harmonically (1) by each of the conics upon the sides of the self-polar triangle (provided they cut the transversal), and (2) by each side of the self-polar triangle and the conic passing through the opposite vertex and the pole of the common chord with respect to the given conic.

4. Four of these conics pass through each of the points of intersection of the sides of the self-polar triangle with the lines joining its vertices to the

pole of their common chord with respect to the given conic, and by joining these points each becomes the vertex of a harmonic pencil.

Some of the results which were proved in connection with the discussion of Fig. 8 are susceptible of the following interpretation, which from a careful examination of the figure may easily be seen to be true.

THEOREM IV. A line drawn through the vertex of any triangle is cut in involution by the circle described upon the opposite side as a diameter, the circumcircle, and the lines joining the feet of the altitudes from the other vertices, a circle whose diameter is the line from this vertex to the intersection of the altitudes and the opposite side, and circles described upon adjacent sides as diameters, and the altitudes upon those sides. The vertex through which this variable transversal is drawn is the center of the involution (and the locus of the foci is a circle with the same center, and a radius equal to a mean proportional between an adjacent side and the distance to the foot of the altitude upon that side).

A line drawn through the intersection of the altitudes is also cut in involution by the three circles upon the sides as diameters, the three circles having for diameters the lines joining this point to the vertices and the sides of the triangle.

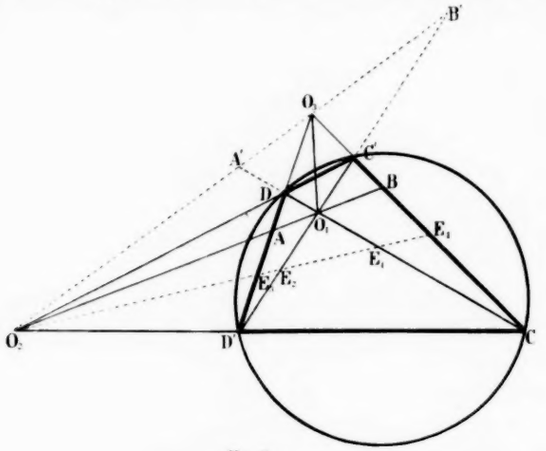

Fig 9

This theorem is also orthogonally and conically projective, excepting the part in the parenthesis.

Let us now resume equation (20) and apply it to the complete quadrilateral of Fig. 9.

Draw any line through O_2 cutting the sides in E_3, E_2, E_1, E_4, and let $O_2E_2 = p$, $O_2E_1 = q$, $O_2E_4 = r$, $O_2E_3 = s$.

Now suppose this line to revolve about O_2 until it passes through O_1. Then E_1, E_2 coincide, $q = p$, and (20) reduces to

$$p\,(r + s) = 2rs\,, \tag{41}$$

whence

$$\frac{r - p}{p - s} = \frac{r}{s}\,, \tag{42}$$

$$\therefore \quad \frac{O_1B}{O_1A} = -\frac{O_2B}{O_2A}\,, \quad \frac{O_2A \,.\, O_1B}{O_1A \,.\, O_2B} = -1\,. \tag{43}$$

By supposing the line to revolve until it passes through O_3 we get in the same way

$$\frac{O_3B'}{O_3A'} = -\frac{O_2B'}{O_2A'}\,, \quad \text{or} \quad \frac{O_2A' \,.\, O_3B'}{O_3A' \,.\, O_2B'} = -1\,. \tag{44}$$

That is, according to our present conception of the quadrilateral, the lines joining the intersections of opposite sides are divided harmonically by the diagonals, and hence at each D-point the two diagonals together with the two sides of the triangle of the D-points form a harmonic pencil.

This furnishes a simple proof of the familiar theorem that in a complete quadrilateral (inscribed in a circle) each diagonal is divided harmonically by the other two. But we have also shown that the relation expressed in (20) is

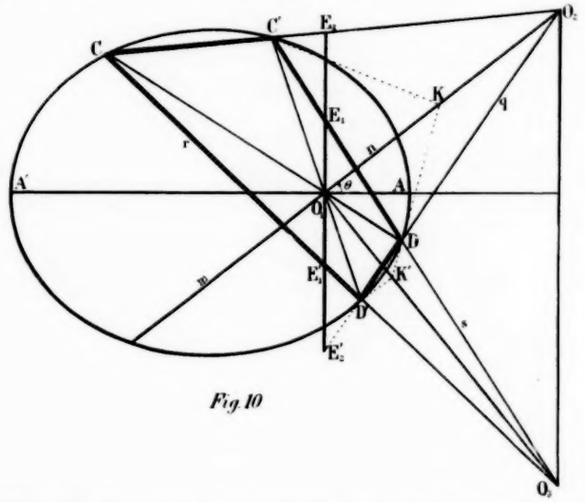

Fig 10

projective, orthogonally, and consequently it is true for any quadrilateral inscribed in an ellipse, that is, for any convex quadrilateral. Therefore, in any complete convex quadrilateral each diagonal is divided harmonically by the other two.

Consider next this equation in its application to a quadrilateral inscribed in an ellipse with the interior D-point at one focus.

Let m, n represent the segments of the variable focal chord, and p, q, r, s, the segments intercepted by the sides upon which these letters are written in Fig. 10. Then, for O_2, (20) takes the form

$$\frac{mn}{m-n} = \frac{pq}{p-q} = \frac{rs}{r-s}. \tag{45}$$

By means of the polar equation of the ellipse it is easily shown that

$$\frac{mn}{m-n} = \frac{pq}{p-q} = \frac{rs}{r-s} = \frac{a(1-e^2)}{2e\cos\theta}, \tag{46}$$

where θ is the angle the segment n makes with the major axis. From this equation, it is seen that these variable ratios are equal to the distance on the variable transversal between the focus and a line perpendicular to the axis at a point midway between the focus and the directrix; for multiplying (46) by two and placing the results equal to ρ gives

$$\rho = \frac{2mn}{m-n} = \frac{2pq}{p-q} = \frac{2rs}{r-s} = \frac{a(1-e^2)}{e\cos\theta}, \tag{47}$$

which is the polar equation of the directrix, the focus being taken as the pole.

When $\theta = 90°$

$$\rho = \frac{2mn}{m-n} = \frac{2pq}{p-q} = \frac{2rs}{r-s} = \infty. \tag{48}$$

$$\therefore \quad m = n,\ p = q,\ r = s, \quad \text{i. e.,} \quad O_1E_1 = O_1E_1',\ O_1E_2 = O_1E_2'. \tag{49}$$

Therefore right lines drawn through the opposite ends of two focal chords of an ellipse intersect the latus rectum at equal distances from the focus.

When q is positive and equal to p and when s is positive and equal to r, we have, respectively,

$$\rho_1 = p = \frac{a(1-e^2)}{e\cos\theta_1}, \quad \text{and} \quad \rho_2 = r = \frac{a(1-e^2)}{e\cos\theta_2}. \tag{50}$$

Hence the opposite sides intersect on the directrix, and since the triangle of the D-points has been shown to be self-polar, it follows that the directrix is the polar of the focus. Of course this result is well known, but it serves as an important check for the validity of this chain of demonstrations.

Also, tangents at the extremities of these focal chords intersect on the sides of the self-polar triangle; and the line joining the focus to the intersec-

tion of two tangents bisects the angle between the focal radii of the points of tangency.*

Therefore the self-polar triangle is right-angled at the focus.

Some interesting consequences can now be deduced by inverting Fig. 10 using the focus as the center of inversion.

We thus obtain the *limaçon* of Fig. 11 with circles through the focus and the adjacent extremities of two focal chords.

Putting m and n for the segments intercepted by the limaçon on a variable focal chord, and p, q, r, s for the segments intercepted by the circles whose centers are marked by these letters, respectively, and remembering that these

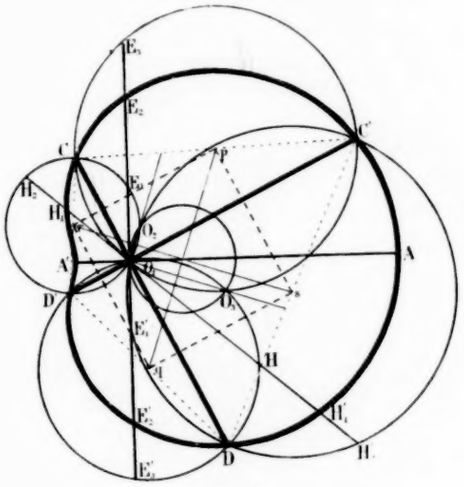

Fig 11

are the reciprocals of the corresponding lines of Fig. 10, we get, by substituting in (47) and reducing and also taking the reciprocal of ρ

$$\rho = \tfrac{1}{2}(n - m) = \tfrac{1}{2}(q - p) = \tfrac{1}{2}(s - r) = \frac{e\cos\theta}{a(1 - e^2)}. \qquad (51)$$

This is the equation of the circle $O_2O_1O_3$, which, as it is the inverse of the directrix, may be called the "Directrix Circle." It will be observed that this result is exactly the same in form as (23).

When $\theta = 90°$, $2\rho = (n - m) = (q - p) = (s - r) = 0$,

$$\therefore \qquad n = m,\ q = p,\ s = r, \quad \text{i. e.,} \quad O_1E_1 = O_1E_1',\ O_1E_3 = O_1E_3'. \qquad (52)$$

* Newcomb's Anal. Geom., p. 166, Ex. 54.

Therefore when circles are drawn through the focus and the ends of two focal chords of a limaçon, opposite circles intersect the parameter at equal distances from the focus.

When the variable focal chord is tangent to one circle, say p, its intercept reduces to zero, and we have

$$n - m = q = s - r, \text{ and } \therefore\ n - q = m,\ s - q = r,\ r - m = s - n, \quad (53)$$

or, as is shown on the figure

$$O_1H_1 = HH_1',\ O_1H_2 = HH_2',\ H_1H_2 = H_1'H_2'.$$

Hence, when the variable focal chord is tangent to one circle, it is intersected by the opposite circle, and also by the other two circles, at equal distances from its intersection with the limaçon.

As p, r in turn become positive and equal to q, s respectively we get from (51)

$$p_1 = q = \frac{e\cos\theta_1}{a(1-e^2)}, \quad p_2 = s = \frac{e\cos\theta_2}{a(1-e^2)}, \quad (54)$$

which shows that opposite circles intersect on the directrix circle. This fact also follows directly, for O_2 and O_3 are the inverse points of the two exterior D-points of Fig. 10, and hence furnishes another check for the truth of our results. Consequently the lines of centers of opposite circles, pq and rs, pass through the center of the directrix circle and are perpendicular to O_1O_3, O_1O_2, respectively; but these last two lines are the sides of the self-polar triangle of Fig. 10, which, as was stated before, are at right angles. Hence pq is perpendicular rs. Also the lines of centers of adjacent circles are perpendicular to the fixed focal chords, since the focal radii are the common chords of these pairs of circles.

Therefore the centers of these four circles are the vertices of a rhombus whose center is the center of the directrix circle.

As a special case of this the following may be noted:

If a quadrilateral is formed by joining the ends of two perpendicular focal chords of a limaçon, the lines joining consecutively the middle points of its sides enclose a square whose center is the center of the directrix circle and whose side is equal to half the parameter. Fig. 11 exemplifies this special case.

In the discussion of Fig. 10, we might have included the other similar ellipses drawn through the focus and the vertices of the quadrilateral; these would invert into cubics (since the center of inversion is on the conics) through the ends of the focal chords of the limaçon. We should thus have, as before,

the variable transversal in Fig. 11 cut in involution by the limaçon, the four circles, and the four cubics. The special results as to concurrence and collinearity would be the same as in former figures. However, one consequence deserves special attention.

In these involutions conjugate points lie respectively on a circle and the opposite cubic ; also the conjugate of the center is at infinity.

Therefore, the tangents to the circles at the focus of the limaçon are the asymptotes of the cubics.

Since the parabola is but the special case of the ellipse, with one focus removed to infinity, it follows that all these results will remain true if we substitute the word " parabola " for " ellipse." For precisely the same reason we may write " cardioid " in the place of " limaçon."

Now suppose that a given circle intersects a given ellipse (or parabola) in four real points. Draw the quadrilateral determined by the four points of intersection. Also draw the three systems of circles as in Fig. 7, and the three systems of similar ellipses (or parabolas) as stated in the theorem derived by orthogonal projection. Let the small letters denote the intercepts of the quadrilateral and the circles as in Fig. 6, and let the corresponding capitals represent the intercepts made by the corresponding conics, that is, conics through the same three points as the circles, respectively.

Then we have for the interior D-point

$$\frac{MN}{M-N} = \frac{mn}{m-n} = \frac{pq}{p-q} = \frac{rs}{r-s}, \tag{55}$$

$$mn = p'q = pq' = r's = rs', \tag{56}$$

$$MN = Pq = pQ = Rs = rS, \tag{57}$$

$$\therefore \quad \frac{mn}{MN} = \frac{p'}{P} = \frac{q'}{Q} = \frac{r'}{R} = \frac{s'}{S} = \text{a variable}. \tag{58}$$

The following conclusions are now apparent :

1. Intercepts made by the given circle, the given conic, opposite sides of the quadrilateral, opposite circles, and opposite conics are simultaneously equal.

2. The intercepts of a circle and a conic through the same three points are proportional to those of any other circle and conic through three common points, and also proportional to the products of the segments made by the given circle and the given conic.

3. For parabolas M (or N), P, Q, R, S become infinite simultaneously.

4. In (58) the terms of any fraction must reduce to zero simultaneously.

Therefore, a circle and a conic through the same three points have a common tangent at the D-point which is parallel to the opposite side of the quadrilateral.

Now conceive the seven circles of Fig. 8 to be drawn on the same diagram, and the same number of similar ellipses (or parabolas) with axes parallel to the axes of the given ellipse (or parabola) to be drawn through the same three points, respectively, as the circles. We thus add four members to each of the equations (56), (57), (58), and extend these relations to the intercepts of nine circles and nine ellipses (or parabolas). Evidently the same equations are true for either of the exterior D-points. The diagram would now contain twenty circles and twenty similar ellipses (or parabolas) with parallel axes, all of which are connected by these relations, as the transversal is drawn through the different D-points. No attempt has been made to draw this figure.

These properties are also orthogonally projective, and by this means we obtain a figure which consists of a group of twenty similar ellipses with parallel axes, and another group of twenty similar ellipses (or parabolas) with parallel axes, but whose axes are not necessarily parallel to those of the first group.

Therefore equations (55) to (58) are true when the quadrilateral is formed by joining the four real points of intersection (1) of two ellipses (2) of an ellipse and a parabola (3) of two parabolas, and the interpretation is the same as that already given for the circle and the ellipse.

When the given conics are both parabolas their axes cannot be parallel. In this case the numerators of (58) become simultaneously infinite as well as the denominators.

As a special case we may suppose the given circle to intersect the given ellipse (or parabola) so that the interior D-point shall coincide with the focus. Then using the focus as center of inversion we get a quadrilateral inscribed in both a circle and a limaçon (or cardioid), a system of four circles through the focus, and a system of four cubics through the vertices of the quadrilateral, inverting only the two systems of curves through the focus of the conic.

The inverse of (55) takes the form

$$M - N = m - n = p' - q' = r' - s' , \tag{59}$$

which should be compared with (23) and (51).

Equations (56), (57) invert respectively into

$$mn = pq' = p'q = rs' = r's , \tag{60}$$

$$MN = Pq' = p'Q = Rs' = r'S , \tag{61}$$

$$\therefore \quad \frac{mn}{MN} = \frac{p}{P} = \frac{q}{Q} = \frac{r}{R} = \frac{s}{S} . \tag{62}$$

The small letters have their usual significance, and M, N denote the segments of the variable transversal made by the quartic, and P, Q, R, S, denote the intercepts of the four cubics.

The meaning of this result is evident.

These results will doubtless suggest to the reader the familiar theorem that "A system of conics passing through four fixed points meets any transversal in a system of points in involution."

The involutions here considered are not special cases of that stated in the theorem though they are closely allied to it, as the centers of the former are coincident conjugate points, and therefore foci, of the latter, when the two transversals coincide.

The subject is not yet exhausted, but the discussion has been carried far enough to illustrate the resources and advantages of the modern geometrical methods.

THE CALCULUS OF VARIATIONS: DERIVATION OF SOME OF THE FUNDAMENTAL WEIERSTRASSIAN FORMULÆ.

By Dr. Harris Hancock, Chicago, Ill.

1. Consider the integral

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt,$$

where F is a given function of the four arguments x, y, x', y', the quantities x' and y' being written for $\frac{dx}{dt}$ and $\frac{dy}{dt}$; farther we must regard F as a *one valued* regular function of these four arguments, one valued not in the analytical sense, but only for real values of the arguments. x and y are defined for the whole plane or for a connected portion of it, while x' and y' are to be considered as variables that are not limited, since they determine the direction of the tangent, and it is supposed that we may go in any direction from the point x, y. In our problem new assumptions are made regarding x and y, but not regarding x' and y'. We further assume that the functions x, y, x' and y' are capable of being differentiated, and that the curve is regular throughout its whole extent, or is composed of regular portions. Consequently x and y considered as functions of t and written $x(t)$, $y(t)$, are one valued regular functions of t throughout its whole extent or throughout the regular portions; in the latter case we shall limit ourselves to one regular portion. If we did not make this assumption, the curve could not be the object of mathematical investigation, since there is no method of treating in their generality irregular curves; and, if we wish the rules of the differential and integral calculus to be sufficient, then we must first apply our investigation to such functions, to which the rules are applicable without any limitation; that is, to functions having the above properties.

2. If we find a curve which is regular and which satisfies the conditions of the problem, then it remains yet as a supplement to prove that it is the only curve which satisfies the condition of the problem.*

For example, it is found that of all regular closed curves of given perimeter, the circle is the one which incloses the greatest surface; *à priori*, however, it is not known that a regular curve satisfies the problem. We know

* Extracts from Weierstrass' lectures on "The Calculus of Variations."

that of all polygons with a given number of sides and having a given perimeter the regular polygon has the greatest surface, and we thus come to the conclusion that the circle, to which the polygon approaches when the number of sides is increased, will have the greatest surface of all the closed curves; however, no one will recognize in this a rigorous proof, and in fact there yet remains a peculiar artifice to prove this property of the circle. We may at the same time point out the difficulty from which the calculus of variations at the present time is not entirely free.

3. The chief difficulty in all analysis consists in giving a strenuous proof that the necessary conditions that have been found for the existence of a certain property are also sufficient. In analytical researches we make conclusions in the following manner: If the analytical quantities exist which are required through the problems that have been set, then they must have certain properties; this gives the necessary conditions for the sought functions. It remains yet reciprocally to prove: if the conditions for an analytical object (curve, surface, etc.) are fulfilled, then the analytical object satisfies the conditions of the problem.

We therefore presuppose in our investigations, that the required functions are regular in their whole extent, and we seek the necessary conditions for the function which are given from the problems. Finally we will free ourselves from the limitations as far as it is possible, and see whether also the functions which have been found correspond to the conditions of the problem.

4. We will now prove an important property of the function F (Art. 1). In all the problems* which have been discussed the following is to be observed: the value of the integral, which is to be a minimum, depends in all cases only upon the form of the curve which is to be determined, not upon the manner in which x, y are represented as functions of a quantity t.

For example, if in the first problem we write the integral in the form

$$\int_{x_0}^{x_1} y \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx,$$

then t is exactly equal to x, and it is clear that the value of this integral is the same as it was for the previous form (p. 183).

If we write for t any function of another quantity τ of such a nature that to the values t_0 and t_1 of t the values τ_0 and τ_1 of τ correspond, and that the curve with increasing τ will be traversed in the same direction as in the first case with increasing t, then the integral must remain unaltered, if it is to be

* See Annals of Mathematics, Vol. IX, p. 180, et seq.

independent of the manner in which x, y are represented as functions of a quantity t; that is, we must have :

1) $$\int_{t_0}^{t_1} F\left[x, y, \frac{dx}{dt}, \frac{dy}{dt}\right] dt = \int_{\tau_0}^{\tau_1} F\left[x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right] d\tau .$$

The simplest function of this kind that we can write for t, is $t = k\tau$, where k represents any arbitrary but *positive* quantity. Hence considering x, y as functions of τ in the left hand side of 1) we have

$$\int_{t_0}^{t_1} F\left[x, y, \frac{dx}{dt}, \frac{dy}{dt}\right] dt = \int_{\tau_0}^{\tau_1} F\left[x, y, \frac{1}{k}\frac{dx}{d\tau}, \frac{1}{k}\frac{dy}{d\tau}\right] k d\tau ;$$

hence

2) $$\int_{\tau_0}^{\tau_1} F\left[x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right] d\tau = \int_{\tau_0}^{\tau_1} F\left[x, y, \frac{1}{k}\frac{dx}{d\tau}, \frac{1}{k}\frac{dy}{d\tau}\right] k d\tau .$$

Since this equation must be true for any arbitrary positive value of k, which however is not necessarily a constant, but may be any continuous positive function, it follows that the functions to be integrated must themselves be equal for every positive value of k ; and consequently

$$F\left[x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right] = kF\left[x, y, \frac{1}{k}\frac{dx}{d\tau}, \frac{1}{k}\frac{dy}{d\tau}\right];$$

or

3) $$F(x, y, kx', ky') = kF(x, y, k', y') .$$

That is, if the integral I is to depend only upon the form of the curve (or, in other words, upon the analytical connection between x and y), then $F(x, y, x', y')$, with regard to x' and y', must be a homogeneous function of the first degree. This condition is also sufficient ; for consider x, y first expressed as function of a quantity t and then as functions of a quantity τ, and if these functions are of such a nature that the curve is traversed from the beginning point to the end point when t takes all values from t_0 and t_1, and τ all values from τ_0 to τ_1, then we can write $\frac{dt}{d\tau} = k$, if t increases at the same time as τ. Since k is a positive quantity, the correctness of the expression 2) follows from the existence of 3) and at the same time also the correctness of 1).

5. The condition that $F(x, y, x', y')$ must be a homogeneous function of the first degree with regard to x' and y' is generally expressed in another manner.

In fact it is nothing else than the condition of integrability of F. For if $F(x, y, x', y')\,dt$ is to be an exact differential, so that say

$$F(x, y, x', y') = d\Phi\,,$$

then the equation

$$F(x, y, x', y') = \frac{\partial \varphi}{\partial x}\,x' + \frac{\partial \varphi}{\partial y}\,y' + \frac{\partial \varphi}{\partial x'}\,x'' + \frac{\partial \varphi}{\partial y'}\,y''$$

must exist identically.

Since no second differential quotient is present in F, it follows that

$$\frac{\partial \varphi}{\partial x'} = 0 \quad \text{and} \quad \frac{\partial \varphi}{\partial y'} = 0\,,$$

and therefore

$$F(x, y, x', y') = \frac{\partial \varphi}{\partial x}\,x' + \frac{\partial \varphi}{\partial y}\,y'\,.$$

But this is nothing more than that F is a homogeneous function of the first degree in x', y'.

This is everywhere the case in the examples given above (pp. 183 et seq.).

6. If the curve is of such a nature that one may regard the one coordinate as a one valued function of the other, and in such a way that for every value of x between two limits x_0 and x_1, there corresponds only one definite value of y, and that x continuously increases when we traverse the curve from the beginning point to the end point, then we may choose for t the quantity x itself, and therefore with the integral in the form :

4)
$$I = \int_{x_0}^{x_1} F\left[x, y, 1, \frac{dy}{dx}\right] dx\,,$$

as it is usually written.

7. This representation is not always true, since the above conditions which are necessary are not always fulfilled ; for example, in the fourth problem (p. 186) we must distribute the yet unknown curve into several parts, which does not appear convenient.

On the other hand, a representation such as given above is always possible, if we introduce the quantity t ; since one could introduce as the variable t the arc s of the curve measured from the beginning point. Besides in the form 4) it sometimes unavoidably happens that $\frac{dy}{dx}$ and consequently F becomes infinite within the limits of integration ; on the other hand it is generally possible so to choose t that this is not the case.

For these reasons we* prefer, in spite of the fact that many developments

* With Prof. Weierstrass.

become more cumbrous, to represent the problem of the calculus of variations in the form:

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt\,;$$

for on the other hand its great symmetry overbalances the fault given above.

8. *Analytical condition for* $F(x, y, x', y')$.

In the relation (Art. 4)

$$F(x, y\,;\, kx', ky') = kF(x, y, x', y')\,,$$

write $k = 1 + h$, then is

$$F(x, y\,;\, (1 + h)\, x', (1 + h)\, y') = (1 + h)\, F(x, y, x', y')$$

or:

$$F + \left\{\frac{\partial F}{\partial x'} x' + \frac{\partial F}{\partial y'} y'\right\} h + h^2(\ldots) = F + hF$$

where

$$F = F(x, y, x', y')\,.$$

Therefore equating the coefficients of h:

1)
$$F = x' \frac{\partial F}{\partial x'} + y' \frac{\partial F}{\partial y'}\,, \ldots, \tag{1}$$

which is another condition of homogeneity which F must have.

9. Differentiate 1) first with regard to x' and then with regard to y', which is allowable, since F is a regular function, and x', y' vary in a continuous manner, and we have

$$\left.\begin{aligned} &\alpha) \quad \frac{\partial^2 F}{\partial x'^2} x' + \frac{\partial^2 F}{\partial y' \partial x'} y' = 0 \\ &\beta) \quad \frac{\partial^2 F}{\partial x' \partial y'} x' + \frac{\partial^2 F}{\partial y'^2} y' = 0 \end{aligned}\right\} \ldots. \tag{2}$$

Hence from α)

$$\frac{\partial^2 F}{\partial x'^2} : \frac{\partial^2 F}{\partial y' \partial x'} = y' : -x' = y'^2 : -x'y'$$

and from β)

$$\frac{\partial^2 F}{\partial x' \partial y'} : \frac{\partial^2 F}{\partial y'^2} = -y' : x' = -x'y' : x'^2\,.$$

Therefore

$$\frac{\partial^2 F}{\partial x'^2} : \frac{\partial^2 F}{\partial y' \partial x'} : \frac{\partial^2 F}{\partial y'^2} = y'^2 : -x'y' : x'^2\,;$$

and if F_1 denotes a factor of proportionality we have:

3)
$$\frac{\partial^2 F}{\partial x'^2} = F_1 y'^2\,;\ \frac{\partial^2 F}{\partial x' \partial y'} = -F_1 x'y'\,;\ \frac{\partial^2 F}{\partial y'^2} = F_1 y'^2\,.$$

And consequently

$$\frac{\partial^2 F}{\partial x'^2}\Big/ y'^2 = \frac{\partial^2 F}{\partial x' \partial y'}\Big/ (-x'y') = \frac{\partial^2 F}{\partial y'^2}\Big/ y'^2 = F_1 .$$

F is of the 1st dimension in x' and y'; $\frac{\partial F}{\partial x'}$, $\frac{\partial F}{\partial y'}$ are of the dimension 0 in x' and y' and $\frac{\partial^2 F}{\partial y'^2}$, $\frac{\partial^2 F}{\partial x' \partial y'}$, $\frac{\partial^2 F}{\partial y'^2}$ are of the — 1st dimension; consequently F_1 is of the dimension — 3 in x' and y'.

10. *A Closer Study of the First Variation.*

We return to the integral (see Art. 1):

1)
$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt ,$$

and make the supposition, that along the curve P_0P_1, the quantities x, y, and

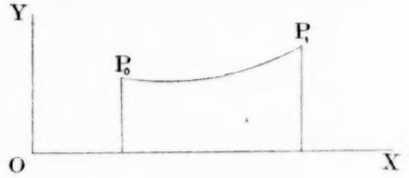

Fig. 1.

their first derivatives with regard to t are continuous functions of t; and we now no longer* assume that the points P_0 and P_1 remain fixed.

We make the substitutions:

$$\begin{aligned}
x &\quad x + \varepsilon\xi + \frac{\varepsilon^2}{1.2}\xi^2 + \dots , \\
y &\quad y + \varepsilon\eta + \frac{\varepsilon^2}{1.2}\eta^2 + \dots , \\
x' &\quad x' + \varepsilon\xi' + \frac{\varepsilon^2}{1.2}\xi'^2 + \dots , \\
y' &\quad y' + \varepsilon\eta' + \frac{\varepsilon^2}{1.2}\eta'^2 + \dots ,
\end{aligned}$$

where ξ, η, ξ', η' are continuous functions of t; and ξ, η are not zero at the points t_0 and t_1.

* Cf. The Annals of Mathematics, Art. 10, p. 185.

When we make the substitution above, the single points on the curve are subjected to a certain *sliding*.

If we neglect the 2nd powers of ε, then each point, owing to the substitution, moves along a straight line ; and this is quite sufficient in the investigations concerning the calculation of the first variation.

11. Making this substitution in 1) we have :

$$I + \Delta I = \int_{t_0}^{t_1} F(x + \varepsilon\xi,\, y + \varepsilon\eta,\, x' + \varepsilon\xi',\, y' + \varepsilon\eta')\, dt\,,$$

which developed by Maclaurin's theorem is

$$= \int_{t_0}^{t_1} \left\{ F + \varepsilon\left[\frac{\partial F}{\partial x}\,\xi + \frac{\partial F}{\partial y}\,.\,\eta + \frac{\partial F}{\partial x'}\,\xi' + \frac{\partial F}{\partial y'}\,\eta'\right] + \varepsilon^2(\ldots)\right\} dt\,.$$

Further (p. 189)

$$\Delta I = \varepsilon\delta I + \frac{\varepsilon^2}{1\,.\,2}\,\delta^2 I + \ldots\,;$$

hence equating coefficients of ε,

2)
$$\delta I = \int_{t_0}^{t_1} \left[\frac{\partial F}{\partial x}\,\xi + \frac{\partial F}{\partial y}\,\eta + \frac{\partial F}{\partial x'}\,\xi' + \frac{\partial F}{\partial y'}\,\eta'\right] dt\,.$$

But

$$\int_{t_0}^{t_1} \frac{\partial F}{\partial x'}\,.\,\frac{d\xi}{dt}\,.\,dt = \left[\frac{\partial F}{\partial x'}\,.\,\xi\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \xi\,\frac{d}{dt}\left[\frac{\partial F}{\partial x'}\right] dt\,;$$

so that 2) becomes

2′)
$$\delta I = \int_{t_0}^{t_1} \left\{ \left[\frac{\partial F}{\partial x} - \frac{d}{dt}\left[\frac{\partial F}{\partial x'}\right]\right]\xi + \left[\frac{\partial F}{\partial y} - \frac{d}{dt}\left[\frac{\partial F}{\partial y'}\right]\right]\eta \right\} dt + \left[\frac{\partial F}{\partial x'}\,.\,\xi + \frac{\partial F}{\partial y'}\,\eta\right]_{t_0}^{t_1};$$

or

3)
$$\delta I = \int_{t_0}^{t_1} \{G_1\xi + G_2\eta\}\, dt + \left[\frac{\partial F}{\partial x'}\,\xi + \frac{\partial F}{\partial y'}\,\eta\right]_{t_0}^{t_1},$$

where

$$G_1 = \frac{\partial F}{\partial x} - \frac{d}{dt}\left[\frac{\partial F}{\partial x'}\right],$$

$$G_2 = \frac{\partial F}{\partial y} - \frac{d}{dt}\left[\frac{\partial F}{\partial y'}\right].$$

12. *On the proof that $G_1 = 0 = G_2$ in order that the integral in question may have a maximum or a minimum value.*

ξ and η are functions of t which we may choose at pleasure. Among all the possible variations we choose those by which the end points P_0 and P_1 remain fixed; and hence at these points $\xi = 0 = \eta$.

We still further assume that the sliding takes place parallel to the axis x, i. e., that $\eta = 0$.

Then choosing

$$\xi = \left[\frac{\partial F}{\partial x} - \frac{d}{dt}\left\{\frac{\partial F}{\partial x'}\right\}\right](t - t_0)(t_1 - t),$$

it is seen that

$$\delta I = \int_{t_0}^{t_1} G_1^2 (t - t_0)(t_1 - t),$$

where G_1 is by supposition continuous.

In this expression for δI, all the terms are positive, so that δI can only be zero* when

$$G_1 = 0$$

along the whole curve.

In a similar manner, when we write

$$\eta = G_2 (t - t_0)(t_1 - t),$$

it is seen that

$$G_2 = 0$$

along the whole curve.

ξ and η must be continuous and also the differential quotients $\frac{d\xi}{dt}, \frac{d\eta}{dt}$ must exist in these investigations; and from our previous methods it is seen that the derivatives of x and y with respect to t up to the third order must exist, because the radius of curvature must constantly change and in a continuous manner.

In the following researches only derivatives up to the third order occur, so that the derivatives of x and y with respect to t need only exist up to this order.

13. *On the reduction of G_1 and G_2 to a single equation.*

We wrote (Art. 11)

$$G_1 = \frac{\partial F}{\partial x} - \frac{d}{dt}\left\{\frac{\partial F}{\partial x'}\right\};$$

and it was shown in Art. 8 that

$$F(x, y, x', y') = x' \frac{\partial F}{\partial x'} + y' \frac{\partial F}{\partial y'}.$$

* Art. 2, p. 82.

Therefore

$$\frac{\partial F}{\partial x} = x' \frac{\partial^2 F}{\partial x' \partial x} + y' \frac{\partial^2 F}{\partial y' \partial x},$$

and

$$-\frac{d}{dt}\left[\frac{\partial F}{\partial x'}\right] = -\frac{\partial^2 F}{\partial x \partial x'} x' - \frac{\partial^2 F}{\partial y \partial x'} y' - \frac{\partial^2 F}{\partial x'^2} \frac{dx'}{dt} - \frac{\partial^2 F}{\partial y' \partial x'} \frac{dy'}{dt}.$$

Hence,

$$G_1 = y' \left[\frac{\partial^2 F}{\partial x \partial y'} - \frac{\partial^2 F}{\partial y \partial x'}\right] - \left[\frac{\partial^2 F}{\partial x'^2} \frac{dx'}{dt} + \frac{\partial^2 F}{\partial y' \partial x'} \frac{dy'}{dt}\right].$$

Writing $\frac{\partial^2 F}{\partial x'^2} = y'^2 F_1$ (see Art. 9), and $\frac{\partial^2 F}{\partial y' \partial x'} = -x'y'F_1$; and defining G by the equation

$$G = \frac{\partial^2 F}{\partial x \partial y'} - \frac{\partial^2 F}{\partial y \partial x'} - F_1\left[y' \frac{dx'}{dt} - x' \frac{dy'}{dt}\right],$$

it is seen that

$$G_1 = y' \,.\, G\,.$$

In a similar manner it may be shown that

$$G_2 = -x' \,.\, G\,.$$

14. One problem was so to determine x and y as functions of t that the integral

$$\int_{t_0}^{t_1} F(x, y, x', y')\, dt,$$

(where F is a given function of the four arguments) should be a maximum or a minimum.

In general $G = 0$ is the differential equation of the problem; however, $G_1 = 0$, $G_2 = 0$, $G = 0$ are three different forms of one and the same differential equation.

From formula 3) Art. 11 we have, when we write $G_1 = y' \,.\, G$ and $G_2 = -x' \,.\, G$,

$$\delta I = \int_{t_0}^{t_1} G\,(y'\xi - x'\eta)\, dt + \left[\xi \frac{\partial F}{\partial x'} + \eta \frac{\partial F}{\partial y'}\right]_{t_0}^{t_1};$$

where along the whole curve it is necessary that $G = 0$, if a maximum or a minimum value of the given integral is to enter.

We thus have a differential equation of the second order for the determination of the required curve.

15. In the differential equation

$$G = 0$$

the second derivatives of the coordinates with respect to t appear only in the simple determinative form:

$$y'x'' - x'y'' .$$

The radius of curvature at any point is:

$$\rho = \frac{\left[1 + \left\{\frac{dy}{dx}\right\}^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\frac{1}{(dx)^3}\, ds^3}{\frac{d}{dx}\left\{\frac{dy}{dt}\middle/\frac{dx}{dt}\right\}} = \frac{\frac{ds^3}{dx^3}}{\frac{y''x' - y'x''}{x'^3}} = \frac{\left\{\frac{ds}{dt}\right\}^3}{y''x' - y'x''};$$

or if k denotes the curvature:

$$y'x'' - x'y'' = -k\left\{\frac{ds}{dt}\right\}^3.$$

Writing this in $G = 0$, it is seen that through the equation $G = 0$ a definite relation is expressed between the curvature of the curve at a definite point, the coordinates, and the direction of the tangent of the curve at this point.

The curve is definitely determined excepting two constants through the differential equation $G = 0$, and the two constants are determined through the two limiting conditions.

16. The equations $G_1 = 0 = G_2$ are often more convenient than $G = 0$; especially is this the case if F does not contain explicitly one of the two quantities x and y, since then, if x, say, is wanting, then $\frac{\partial F}{\partial x} = 0$, and from $G_1 = 0$

$$\frac{d}{dt}\left\{\frac{\partial F}{\partial x'}\right\} = 0,$$

an equation which is at once integrable.

17. We will now free ourselves from some of the previous limiting assumptions. First of all, with regard to the choice of the special variations of the curve, it is clear that owing to this choice the generality of the derived condition, which is necessary for the existence of a maximum or of a minimum, does not suffer any loss; since ΔI is for all variations to be continuous and > 0 (or < 0) and under these variations our special variations are contained, so that the condition that has been found is certainly necessary, however the variations may have been chosen.

We have further assumed that x', y', i. e., the direction of the curve, experiences no sudden changes ; and therefore the curve is to consist of one regular trace. This assumption we now drop, and only retain the assumption that the curve is composed of regular portions, so that consequently the direction of the tangent need not change in a continuous manner at all places. Then each portion of curve must satisfy the differential equation.

18. Since for example, if the curve AB consists of two such portions AC and CB, then among all the possible variations of AC, there are also these, by

Fig. 2.

which CB remains unaltered and only AC is subject to variation.

Now the variation of the integral depends only upon the variation of AC, and we conclude as above, that this portion of the curve must satisfy the differential equation $G = 0$. The same is true of CB ; therefore, along the whole curve the equation $G = 0$ must be fulfilled.

In the case of such a transition from one regular portion to another there remains yet tor the integration of $G = 0$ a very important theorem, the necessity of which we will make clear by means of an example, especially because in the text-books it is overlooked.

Grant that F (as in the first problem, p. 183) is independent of x, and for this reason we will make use of $G_1 = 0$ instead of $G = 0$.

From the equation :

A) $$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial x'} = 0 ,$$

we have here at once

B) $$\frac{\partial F}{\partial x'} = \text{const (with respect to } t) .$$

However, it is not at all excluded, that $\frac{\partial F}{\partial x'}$ makes anywhere a sudden spring, since we do not know whether this function is continuous, and whether we can, therefore, go from the equation A) to B) in the sense that the constant has everywhere the same value.

One would most naturally expect, since x', y' make springs between t_0 and

t_1, that also $\frac{\partial F}{\partial x'}, \frac{\partial F}{\partial y'}$ will be discontinuous, and therefore the constant in different intervals would have a different value.

Hence the more important is the following theorem: *Even if x', y', and therefore the direction of the curve, suffers an alteration which is due to a sudden spring, nevertheless $\frac{\partial F}{\partial x'}, \frac{\partial F}{\partial y'}$, change in a continuous manner along the whole curve.*

A discontinuity could only take place at such a point of transition where x', y' suffer a spring.

19. If to the value t_1 of t there corresponds such a spring, then on both

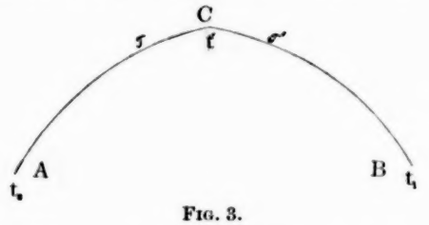

FIG. 3.

sides of t' we take 2 points τ and τ' in such a manner that within the portions $\tau \ldots t'$ and $t' \ldots \tau'$, there is no other discontinuity in the direction of the curve. Then a possible variation of the curve is also the one by which $t_0 \ldots \tau$ and $\tau' \ldots t_1$ remain unaltered, and only the portion $\tau \ldots \tau'$ is varied. Here τ and τ' remain fixed while t_1 is subjected to any kind of sliding.

The variation of the integral,

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt$$

then depends only upon the variation of the integrals

1)
$$\int_{\tau}^{t'} F(x, y, x', y')\, dt + \int_{t'}^{\tau'} F(x, y, x', y')\, dt\,.$$

And in a manner similar to that employed above, the 1st variation of 1) must vanish, so that we have the equation:

$$0 = \int_{\tau}^{t'} G\,(y'\xi - x'\eta)\, dt + \int_{t'}^{\tau'} G\,(y'\xi - x'\eta)\, dt$$
$$+ \left[\frac{\partial F}{\partial x'}\,\xi + \frac{\partial F}{\partial y'}\,\eta\right]_{\tau}^{t'} + \left[\frac{\partial F}{\partial x'}\,\xi + \frac{\partial F}{\partial y'}\,\eta\right]_{t'}^{\tau'}.$$

But since G must vanish along the whole curve, as we have already seen, this equation reduces to

2) $$\left[\frac{\partial F}{\partial x'}\xi + \frac{\partial F}{\partial y'}\eta\right]_\tau^{t'} + \left[\frac{\partial F}{\partial x'}\xi + \frac{\partial F}{\partial y'}\eta\right]_{t'}^{\tau'} = 0\,.$$

For the values τ and τ', they being now supposed belonging to fixed points, the expressions on the left vanish, but we do not know but that each of the two quantities $\frac{\partial F}{\partial y'}$, $\frac{\partial F}{\partial x'}$ have different values at t' according as we approach the point t' from τ or from τ'.

We suppose that this is the case, and consequently we make a distinction between

$$\left[\frac{\partial F}{\partial x'}\right]_{t'}^{-} \text{ and } \left[\frac{\partial F}{\partial x'}\right]_{t'}^{+}, \text{ also between } \left[\frac{\partial F}{\partial y'}\right]_{t'}^{-} \text{ and } \left[\frac{\partial F}{\partial y'}\right]_{t'}^{+}.$$

Then equation 2) becomes

$$\left\{\left[\frac{\partial F}{\partial x'}\right]_{t'}^{-} - \left[\frac{\partial F}{\partial x'}\right]_{t'}^{+}\right\}(\xi)' + \left\{\left[\frac{\partial F}{\partial y'}\right]_{t'}^{-} - \left[\frac{\partial F}{\partial y'}\right]_{t'}^{+}\right\}(\eta)' = 0$$

where $(\xi)'$ and $(\eta)'$ are the values of ξ and η at the point t'. But since these values are quite arbitrary it follows that their coefficients in the above expression must respectively vanish, so that:

3) $$\left[\frac{\partial F}{\partial x'}\right]_{t'}^{-} = \left[\frac{\partial F}{\partial x'}\right]_{t'}^{+} \text{ and } \left[\frac{\partial F}{\partial y'}\right]_{t'}^{-} = \left[\frac{\partial F}{\partial y'}\right]_{t'}^{+};$$

that is, *the quantities* $\frac{\partial F}{\partial x'}$ and $\frac{\partial F}{\partial y'}$ *vary by the transition from the one regular part of the curve to the other in a continuous manner, even if* x' *and* y' *at this point make springs.*

Remark. For the special case that was given above as an example, in which F is independent of x, we conclude at once, that the constant of the differential equation, $\frac{\partial F}{\partial x} = \text{const.}$, has everywhere the same value.

This theorem may be often used to prove that the direction of the curve in question nowhere changes in a discontinuous manner, that the curve does not consist of several regular portions, but of one single regular trace.

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